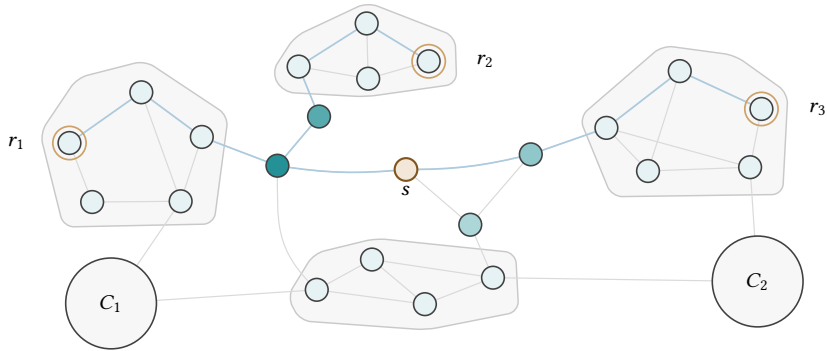
Parallel Small Vertex Connectivity

in Near-Linear Work and Polylogarithmic Depth



Yonggang Jiang
MPI-INF and Saarland University

Changki Yun
University of Maryland

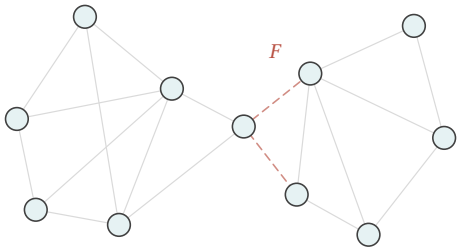
FOCS 2026

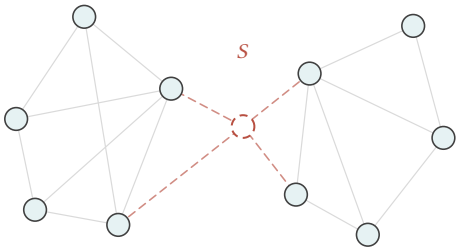
Edge and vertex connectivity

$$\text{Edge connectivity} := \min\{|F| : G - F \text{ is disconnected}\},$$

$$\text{Vertex connectivity} := \min\{|S| : G - S \text{ is disconnected}\}.$$

	Edge connectivity	Vertex connectivity
Sequential	$\widetilde{O}(m)$ [Kar00]	$m^{1+o(1)}$ [LNP+25, CKL+25]
Parallel	$\widetilde{O}(m)$ work, $\widetilde{O}(1)$ depth [GG21, LMN21, AB23]	Can we match these parallel bounds?





Parallel *small* vertex connectivity

Given k, find a cut $|S| < k$ or certify k-connectivity.

Range of k	Work	Depth	Consequence / reference
Arbitrary k	$\widetilde{O}(n^\omega)$	$\widetilde{O}(1)$	[BJMY25]
$k = \text{polylog}(n)$ (dense graphs)	$\widetilde{O}(n^2k^4)$	$\widetilde{O}(k^2)$	[CKT93]
$k < 4$	$\widetilde{O}(m)$	$\widetilde{O}(1)$	[SV82, TV85, MR92]
Arbitrary k	$\widetilde{O}(m)$	$\widetilde{O}(1)$	Implies dense directed reachability of $\widetilde{O}(n^2)$ work, $\widetilde{O}(1)$ depth [BJMY25]
$k = n^{\Theta(1)}$	$\widetilde{O}(m)$	$\widetilde{O}(1)$	Implies "direct sum" of dense reachability of $\widetilde{O}(nk)$ work, $\widetilde{O}(1)$ depth [This work]
$k = \text{polylog}(n)$	$\widetilde{O}(m)$	$\widetilde{O}(1)$	Randomized PRAM, w.h.p. [This work]

Warm-up: (s,t) k-connectivity in $\widetilde{O}(m\,\mathrm{poly}(k))$ work, $\widetilde{O}(\mathrm{poly}(k))$ depth

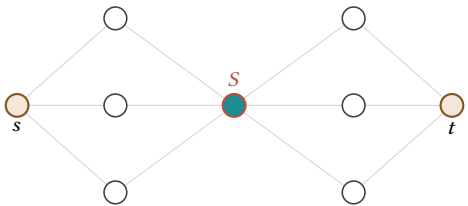
Can undirected shortest paths replace directed searches?

Vertex weights $w \geq 0$, $\quad w(s) = w(t) = 0$, $\quad \sum_v w(v) = 1$.

A cut $|S| < k$

$\exists w$: $\quad \mathrm{dist}_w(s,t) \geq 1/(k-1)$ (Fractional cut)

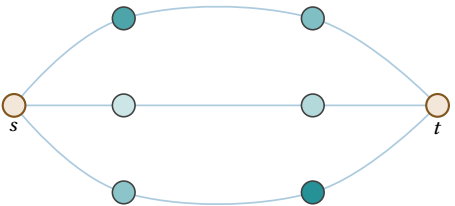
Take $w \propto \mathbf{1}_S$.



No cut $|S| < k$

k internally disjoint paths (Menger)

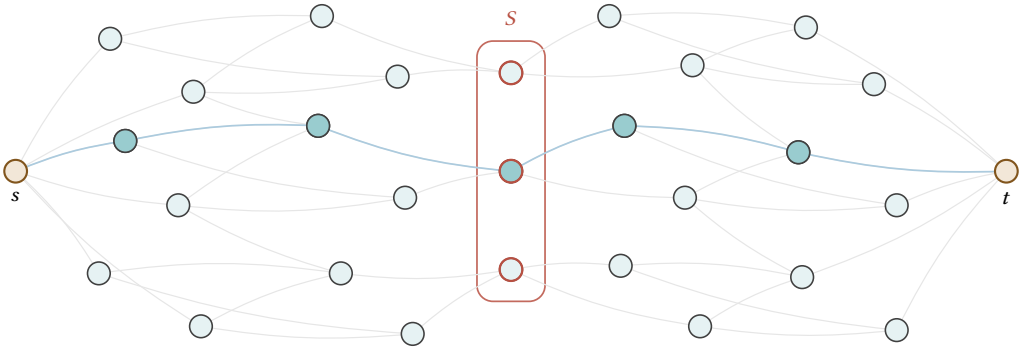
$\forall w$: $\quad \mathrm{dist}_w(s,t) \leq 1/k$



Finding the weights with MWU [GK07, BGS21]

Starting from uniform weights, increase weights along a shortest path P.

$w(v) \leftarrow (1 + 1/\text{poly}(k))w(v)$ for $v \in P \setminus \{s, t\}$.

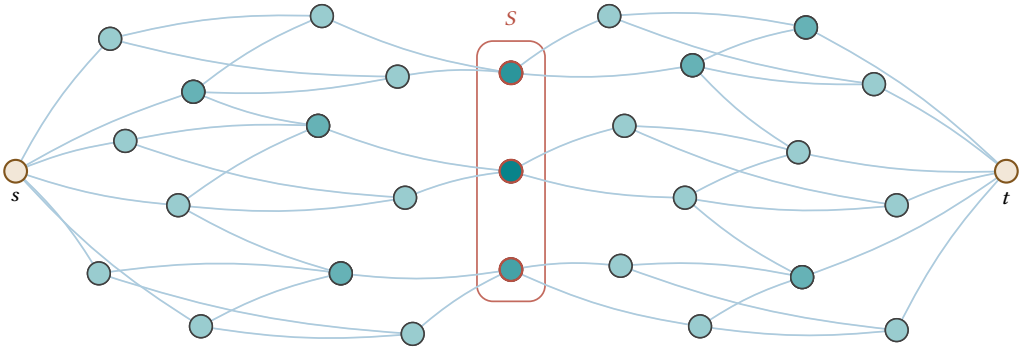


Finding the weights with MWU [GK07, BGS21]

Starting from uniform weights, increase weights along a shortest path P.

If $|S| < k$, MWU finds a fractional cut in $T = \widetilde{O}(\text{poly}(k))$ rounds.

Also stands for $(1 + 1/\Theta(k))$-approximate SSSP – $\widetilde{O}(mk^2)$ work, $\widetilde{O}(k^2)$ depth.[RGH+22].



From (s, t) to global connectivity

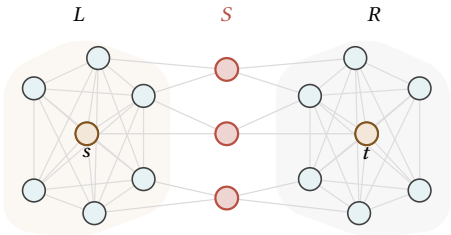
Fix a cut (L, S, R) with $|S| < k$.

Sample s, t independently by degree, with $\mathrm{vol}(A) = \sum_{v \in A} \deg_G(v)$.

$\mathrm{vol}(L), \mathrm{vol}(R) = \Theta(m)$ (Balanced)

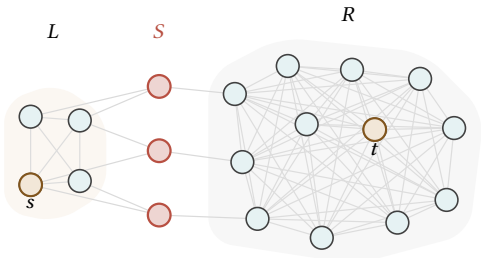
$\Pr[s \in L,\ t \in R] = \Omega(1)$

Reduce to (s, t) k-connectivity.



$\mathrm{vol}(L) \ll \mathrm{vol}(R)$ (Unbalanced)

What if one side has small volume?

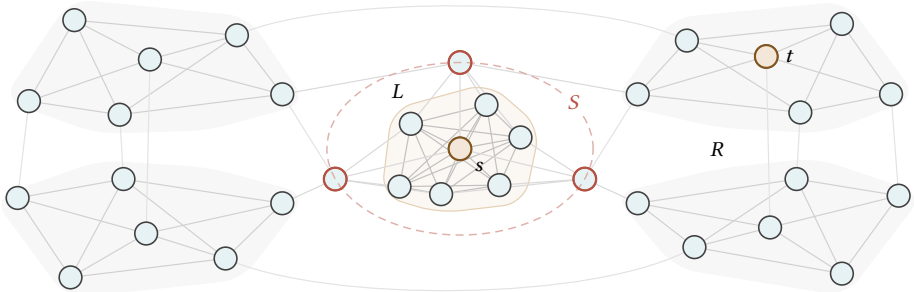


Localize each search

We can still sample (s, t) independently by degree; Suppose $\mathrm{vol}(L) \in [\mu, 2\mu)$.

Sample $\widetilde{O}(m/\mu)$ pairs to hit $L \times R$ with constant probability. [FNY+20]

Localize each search to $\widetilde{O}(\mu\,\mathrm{poly}(k))$ work, instead of naive $\widetilde{O}(m\,\mathrm{poly}(k))$.



Localize the search and its work

Theorem (this work, informal)

After shared preprocessing, randomized LocalCut(s, μ, k) uses

$$\widetilde{O}(\mu \operatorname{poly}(k)) \text{ work} \qquad \widetilde{O}(\operatorname{poly}(k)) \text{ depth.}$$

If a cut (L, S, R) satisfies

$$s \in L, \qquad \operatorname{vol}_G(L) \in [\mu, 2\mu), \qquad |S| < k,$$

it returns a fractional (s, t)-cut of size $< k$ (separating weights) for some t, with constant probability.

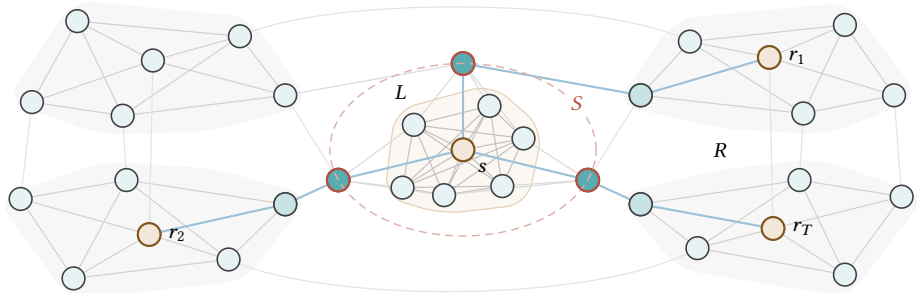
If G is k-connected, it always returns $\perp$.

A separate rounding algorithm converts the fractional cut to an integral cut S. (Omitted)

Localize the MWU search

The target t can change between rounds $1, \ldots, T$.

Any sequence $r_1, \ldots, r_T \in R$ works, since every path still crosses S.



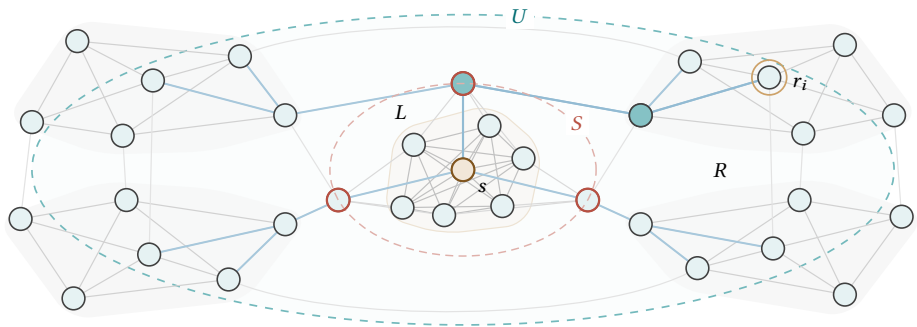
Localize by random termination

Sample r_i from a distance ball U of volume $\widetilde{O}(kT\mu)$.

Assuming $\Delta(G) = O(\mu)$, $\Pr[r_i \in R] = 1 - O(1/T) \implies \Pr[\{r_1, \cdots, r_T\} \subseteq R] = \Omega(1)$.

$\mathrm{dist}_w(s, r_i) > 1/k \implies$ stop with a cut.

Otherwise, update the s–r_i path.

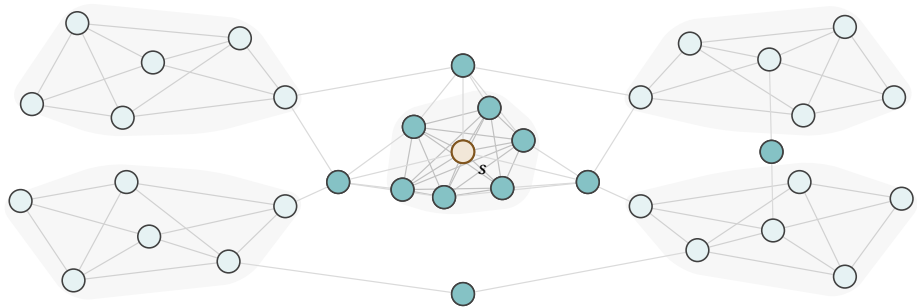


Localize the work

Random termination localizes the search to U.

Finding U by SSSP on G still costs $\widetilde{O}(m\,\mathrm{poly}(k))$ work.

Across all T rounds, updated vertices have total degree $\widetilde{O}(\mu\,\mathrm{poly}(k))$.

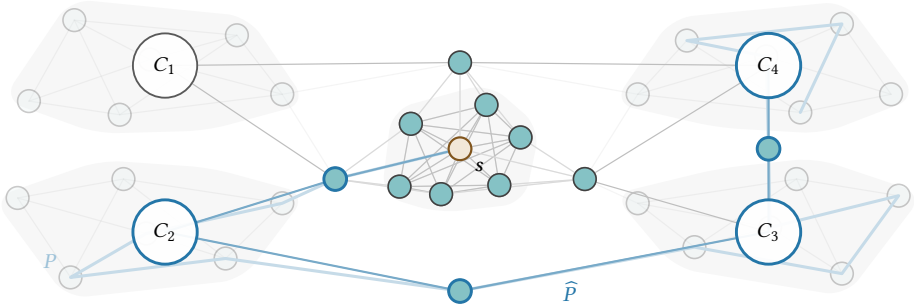


Localize the graph by contraction

Keep updated vertices explicit and contract each weight-1 component C_i.

The contracted graph $\widehat{G}$ has $\widetilde{O}(\mu\,\mathrm{poly}(k))$ edges.

Contraction incurs an *additive* loss of at most n in path length.



Localize with a single-source length sparsifier

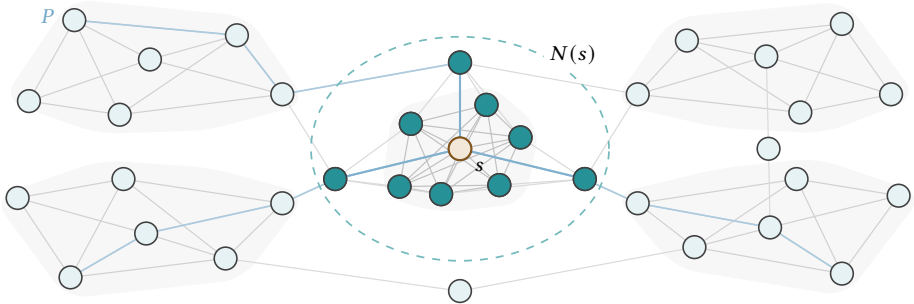
Biased init! $w_{\text{init}}(v) \leftarrow n^3$ for $v \in N(s)$.

Only $O(1)$ overhead in # of MWU rounds.

Every path P leaving s satisfies $w(P) \geq n^3$.

$(1 - n^{-2})w(P) \leq \widehat{w}(\widehat{P}) \leq w(P)$.

SSSP and maintenance fit within $\widetilde{O}(\mu \operatorname{poly}(k))$ total work.



Takeaway

Localize the search with random termination and undirected shortest paths.

Localize the work with a single-source length sparsifier.

$$\widetilde{O}(m\,\mathrm{poly}(k))\text{ work} \qquad \widetilde{O}(\mathrm{poly}(k))\text{ depth}$$

For $k = n^{o(1)}$, this gives $mn^{o(1)}$ work and $n^{o(1)}$ depth.

Our direct-sum barrier through dense directed reachability (DDR)

For polynomial $k = n^{\alpha}$, matching $mn^{o(1)}$ work and $n^{o(1)}$ depth would compute

$$\bigwedge_{i=1}^{n^{1-\alpha}} \mathrm{DDR}(G_i), \qquad |V(G_i)| = n^{\alpha}$$

in $n^{1-\alpha} \cdot (n^{\alpha})^2 \cdot n^{o(1)}$ total work and $n^{o(1)}$ depth.

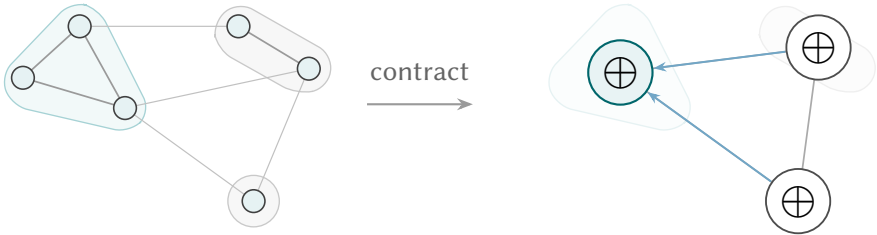
To us, this is as surprising as a breakthrough for a single DDR instance.

One algorithm for PRAM and CONGEST?

The CONGEST result and parallel question of [JM23] motivated this work.

Our PRAM algorithm is pretty different from their technique. *Can we subsume both?*

Minor-Aggregation



Efficient simulations in PRAM and CONGEST [RGH+22].

k-vertex connectivity in $\text{poly}(k, \log n)$ MA rounds?

Thank you



Extended slides

tamref.github.io/talks